(2)

3.4.

$$\frac{\partial}{\partial \tau}\langle \varepsilon \rangle = 2\tau \frac{\partial}{\partial \tau} \ln Z + \tau^2 \frac{\partial^2}{\partial \tau^2} \ln Z$$

$$= \frac{2}{\tau}\langle \varepsilon \rangle + \tau^2 \left[ \frac{\partial^2}{\partial \tau^2} \ln Z \right].$$

$$\frac{\partial^2}{\partial \tau^2} \ln Z = \frac{\partial}{\partial \tau}\left[ \frac{\frac{\partial}{\partial \tau} Z}{Z} \right]$$

$$= \frac{\frac{\partial^2}{\partial \tau^2} Z}{Z} - \frac{\left(\frac{\partial}{\partial \tau} Z\right)\left(\frac{\partial}{\partial \tau} Z\right)}{Z^2}$$

$$= \frac{\frac{\partial^2}{\partial \tau^2} Z}{Z} - \left[ \frac{\partial}{\partial \tau} \ln Z \right]^2.$$

$$\frac{\partial^2}{\partial \tau^2} Z = \sum_s \frac{\partial^2}{\partial \tau^2} e^{-\varepsilon_s / \tau}$$

$$= \sum_s \frac{\partial}{\partial \tau}\left[ \frac{\varepsilon_s}{\tau^2} e^{-\varepsilon_s/\tau} \right]$$

$$= \sum_s -\frac{2\varepsilon_s}{\tau^3} e^{-\varepsilon_s/\tau} + \frac{\varepsilon_s^2}{\tau^4} e^{-\varepsilon_s/\tau}.$$

$$\Rightarrow \quad \frac{\frac{\partial^2}{\partial \tau^2} Z}{Z} = \frac{\langle \varepsilon^2 \rangle}{\tau^4} - \frac{2\langle \varepsilon \rangle}{\tau^3}.$$

$$\frac{2}{\tau}\langle \varepsilon \rangle + \tau^2 \left[ \frac{\partial^2}{\partial \tau^2} \ln Z \right] = \frac{2}{\tau}\langle \varepsilon \rangle + \frac{\langle \varepsilon^2 \rangle}{\tau^2} - \frac{2\langle \varepsilon \rangle}{\tau} + \langle \varepsilon \rangle \frac{\tau}{\tau} - \frac{\langle \varepsilon \rangle^2}{\tau^2}.$$

$$= \frac{1}{\tau^2}\left[ \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 \right]$$

$$\Rightarrow \tau^2 \left(\frac{\partial \langle \varepsilon \rangle}{\partial \tau}\right) = \tau^2 \frac{\partial}{\partial \tau}\langle \varepsilon \rangle = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2$$

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