

Q.

Ratio TP

$$3.4. \quad \frac{d}{dT} \langle \epsilon \rangle = 2T \frac{d}{dT} \ln Z + T^2 \frac{d^2}{dT^2} \ln Z$$

$$= \frac{2}{T} \langle \epsilon \rangle + T^2 \left[\frac{d^2}{dT^2} \ln Z \right].$$

$$\frac{d^2}{dT^2} \ln Z = \frac{d}{dT} \left[\frac{\frac{d}{dT} Z}{Z} \right]$$

$$= \frac{\frac{d^2}{dT^2} Z}{Z} - \frac{\left(\frac{d}{dT} Z \right) \left(\frac{d}{dT} Z \right)}{Z^2}$$

$$= \frac{\frac{d^2}{dT^2} Z}{Z} - \left[\frac{d}{dT} \ln Z \right]^2.$$

$$\frac{d^2}{dT^2} Z = \sum_s \frac{d^2}{dT^2} e^{-\epsilon_s / T}$$

$$= \sum_s \frac{d}{dT} \left[\frac{\epsilon_s}{T^2} e^{-\epsilon_s / T} \right]$$

$$= \sum_s \left[-2 \frac{\epsilon_s}{T^3} e^{-\epsilon_s / T} + \frac{\epsilon_s^2}{T^4} e^{-\epsilon_s / T} \right].$$

$$\Rightarrow \frac{\frac{d^2}{dT^2} Z}{Z} = \frac{\langle \epsilon^2 \rangle}{T^4} - \frac{2 \langle \epsilon \rangle}{T^3}.$$

$$\frac{d}{dT} \langle \epsilon \rangle + T^2 \left[\frac{d^2}{dT^2} \ln Z \right] = \frac{2}{T} \langle \epsilon \rangle + \frac{\langle \epsilon^2 \rangle}{T^2} - \frac{2 \langle \epsilon \rangle}{T} - \frac{\langle \epsilon \rangle^2}{T^2}$$

$$= \frac{1}{T^2} \left[\langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 \right]$$

$$\Rightarrow T^2 \left(\frac{d^2 \ln Z}{dT^2} \right) = T^2 \frac{d}{dT} \langle \epsilon \rangle = \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2$$